

EXOTIC  $n$ -D'ALEMBERT PDE'S AND STABILITY

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**ABSTRACT.** In the framework of the PDE's algebraic topology, previously introduced by A. Prástaro, *exotic  $n$ -d'Alembert PDE's* are considered. These are  $n$ -d'Alembert PDE's,  $(d'A)_n$ , admitting Cauchy manifolds  $N \subset (d'A)_n$  identifiable with exotic spheres, or such that  $\partial N$ , can be exotic spheres. For such equations local and global existence theorems and stability theorems are obtained.

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## 1. INTRODUCTION

*"Do exotic PDE's exist,  
where exotic 7-spheres of the same  $\Theta_7$ -class,  
do not bound smooth solutions?"*

In some previous works we studied  $n$ -d'Alembert PDE's by using the PDE's algebraic topology, introduced by A. Prástaro. (See Refs. [22, 25, 28, 31, 40, 41].) In particular, in [31] are characterized also the stability properties of such equations, showing that the  $n$ -d'Alembert equation is an extended crystal PDE, for any  $n \geq 2$ , and obtaining criteria in order to be an extended 0-crystal PDE and a 0-crystal PDE. Furthermore, we proved that for any  $n \geq 2$  one can canonically associate to the  $n$ -d'Alembert equation another PDE, *stable extended crystal  $n$ -d'Alembert PDE*, having the same regular smooth solutions of the  $n$ -d'Alembert equation, but there, in these solutions, do not occur finite times unstabilities. This allowed to avoid all the problems present in the applications, related to finite unstability of solutions. Furthermore, we formulated a workable criterion to recognize asymptotic stability suitably averaging perturbations. (See [28, 29, 30, 31, 32].)

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As for higher dimensions, i.e., when  $n \geq 7$ , existence of exotic spheres are admitted, it becomes interesting to investigate which implications such phenomena have on the characterization of global solutions of  $n$ -d'Alembert PDE's and their stability. In some previous papers, A. Prástaro has studied in some details such phenomena for the Ricci flow equation, since this is important to prove the Poincaré's conjecture on three dimensional Riemannian manifolds, and its generalizations to higher dimensions. (See [25, 33, 34, 35, 36, 37, 38].) Furthermore, in [39] generalizations of such phenomena are considered for any PDE and characterized in the framework of Prástaro's PDE's algebraic topology.

In this paper we aim to apply this theory to exotic  $n$ -d'Alembert PDE's, and to study the interplay between the geometric stability characterization of such equations by using the algebraic topological methods previously introduced in [28, 29, 30, 31, 32, 35]. (See also [1, 2, 42].)

After this Introduction, the paper splits into two more sections. The first devoted to the characterization of exotic  $n$ -d'Alembert PDE's, and the second to the stability properties of such equations. The main new result is Theorem 3.14 characterizing global solutions of exotic 8-d'Alembert equation. This theorem allows us to answer in the affirmative to the question put in quotation marks, at the beginning of this Introduction. In fact, after Theorem 3.14 we can state that two diffeomorphic exotic 7-sphere, identified with two Cauchy manifolds in  $(d'A)_8$  over  $\mathbb{R}^8$ , bound singular solutions only - they cannot bound smooth solutions. (Compare with the situation in the Ricci flow equation on compact, simply connected 7-dimensional Riemannian manifolds [38].)

## 2. EXOTIC $n$ -D'ALEMBERT PDE's

In this section we resume some our recent results about the algebraic topology characterization of PDE's, and that will be useful in the next section.<sup>1</sup> In particular let us recall the following theorem that relates integral bordism groups of PDE's to subgroups of crystallographic groups. For their proofs we address reader to the original papers.

**Remark 2.1.** *Here and in the following we shall denote the boundary  $\partial V$  of a compact  $n$ -dimensional manifold  $V$ , split in the form  $\partial V = N_0 \cup P \cup N_1$ , where  $N_0$  and  $N_1$  are two disjoint  $(n-1)$ -dimensional submanifolds of  $V$ , that are not necessarily closed, and  $P$  is another  $(n-1)$ -dimensional submanifold of  $V$ . For example, if  $V = S \times I$ , where  $I \equiv [0, 1] \subset \mathbb{R}$ , one has  $N_0 = S \times \{0\}$ ,  $N_1 = S \times \{1\}$ ,  $P = \partial S \times I$ . In the particular case that  $\partial S = \emptyset$ , one has also  $P = \emptyset$ . Let us also recall that with the term quantum solutions we mean integral bordisms relating Cauchy hypersurfaces of  $E_{k+s}$ , contained in  $J_n^{k+s}(W)$ , but not necessarily contained into  $E_{k+s}$ . (For details see [20, 21, 22, 23, 24, 25, 26, 27, 28].)*

**Theorem 2.2.** [35] *Bordism groups relative to smooth manifolds can be considered as extensions of subgroups of crystallographic groups.*

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<sup>1</sup>For general informations on bordism groups, and related problems in differential topology and PDE's geometry, see, e.g., Refs.[3, 7, 8, 9, 10, 11, 13, 20, 21, 22, 23, 24, 25, 26, 27, 29, 44, 45, 47, 48, 49]. For crystallographic groups see references quoted in [35]. About differential structures and algebraic topology of exotic spheres, see [4, 5, 6, 14, 15, 16, 17, 18, 19, 33, 37, 38, 39, 43].

**Definition 2.3.** We say that a PDE  $E_k \subset J_n^k(W)$  is an extended 0-crystal PDE, if its integral bordism group is zero.<sup>2</sup>

The following theorem relates the integrability properties of a PDE to crystallographic groups.

**Theorem 2.4.** (Crystal structure of PDE's).[35] Let  $E_k \subset J_n^k(W)$  be a formally integrable and completely integrable PDE, such that  $\dim E_k \geq 2n + 1$ . Then its integral bordism group  $\Omega_{n-1}^{E_k}$  is an extension of a subgroup of some crystallographic group. In this case, we say that  $E_k$  is an extended crystal PDE and we define crystal group of  $E_k$  the littlest of such crystal groups. The corresponding dimension will be called crystal dimension of  $E_k$ .

Furthermore if  $W$  is contractible, then  $E_k$  is an extended 0-crystal PDE, when  $\Omega_{n-1} = 0$ .

In the following we relate crystal structure of PDE's to the existence of global smooth solutions, identifying an algebraic-topological obstruction.

**Theorem 2.5.** [35, 21, 22, 23]. Let  $E_k \subset J_n^k(W)$  be a formally integrable and completely integrable PDE. Then, in the algebra  $\mathbf{H}_{n-1}(E_k) \equiv \text{Map}(\Omega_{n-1}^{E_k}; \mathbb{R})$ , (Hopf algebra of  $E_k$ ), there is a subalgebra, (crystal Hopf algebra) of  $E_k$ . On such an algebra we can represent the algebra  $\mathbb{R}^{G(d)}$  associated to the crystal group  $G(d)$  of  $E_k$ . (This justifies the name.) We call crystal conservation laws of  $E_k$  the elements of its crystal Hopf algebra.<sup>3</sup>

**Theorem 2.6.** [35, 29, 30, 31, 32]. Let  $E_k \subset J_n^k(W)$  be a formally integrable and completely integrable PDE. Then, the obstruction to find global smooth solutions of  $E_k$  can be identified with the quotient  $\mathbf{H}_{n-1}(E_\infty)/\mathbb{R}^{\Omega_{n-1}}$ .

We define crystal obstruction of  $E_k$  the above quotient of algebras, and put:  $\text{cry}(E_k) \equiv \mathbf{H}_{n-1}(E_\infty)/\mathbb{R}^{\Omega_{n-1}}$ . We call 0-crystal PDE one  $E_k \subset J_n^k(W)$  such that  $\text{cry}(E_k) = 0$ .<sup>4</sup>

**Corollary 2.7.** Let  $E_k \subset J_n^k(W)$  be a 0-crystal PDE. Let  $N_0, N_1 \subset E_k$  be two initial and final Cauchy data of  $E_k$  such that  $X \equiv N_0 \sqcup N_1 \in [0] \in \Omega_{n-1}$ . Then there exists a smooth solution  $V \subset E_k$  such that  $\partial V = X$ .

**Definition 2.8** (Exotic PDE's). Let  $E_k \subset J_n^k(W)$  be a  $k$ -order PDE on the fiber bundle  $\pi : W \rightarrow M$ ,  $\dim W = m + n$ ,  $\dim M = n$ . We say that  $E_k$  is an exotic PDE if it admits Cauchy integral manifolds  $N \subset E_k$ ,  $\dim N = n - 1$ , such that one of the following two conditions is verified.<sup>5</sup>

- (i)  $\Sigma^{n-2} \equiv \partial N$  is an exotic sphere of dimension  $(n-2)$ , i.e.  $\Sigma^{n-2}$  is homeomorphic to  $S^{n-2}$ , ( $\Sigma^{n-2} \approx S^{n-2}$ ) but not diffeomorphic to  $S^{n-2}$ , ( $\Sigma^{n-2} \not\approx S^{n-2}$ ).
- (ii)  $\emptyset = \partial N$  and  $N \approx S^{n-1}$ , but  $N \not\approx S^{n-1}$ .

<sup>2</sup>Here for integral bordism group we refer to weak integral bordism group  $\Omega_{n-1,w}^{E_k}$ .

<sup>3</sup>Recall that  $A \equiv \text{Map}(\Omega, \mathbb{R})$ ,  $\Omega$  a group, has a natural structure of Hopf algebra if  $\Omega$  is a finite group. If  $\Omega$  is not finite group, then  $A$  has a structure of Hopf algebra in extended sense. (See ref.[23].)

<sup>4</sup>An extended 0-crystal PDE  $E_k \subset J_n^k(W)$  does not necessitate to be a 0-crystal PDE. In fact, in order that  $E_k$  is an extended 0-crystal PDE it is enough  $\Omega_{n-1,w}^{E_k} = 0$ . This does not necessarily imply that  $\Omega_{n-1}^{E_k} = 0$ .

<sup>5</sup>In this paper we will use the same notation adopted in [38]:  $\approx$  homeomorphism;  $\cong$  diffeomorphism;  $\simeq$  homotopy equivalence;  $\simeq$  homotopy.

**Example 2.9.** *The Ricci flow equation can be an exotic PDE for  $n$ -dimensional Riemannian manifolds of dimension  $n \geq 7$ . (See [38].)*

**Example 2.10.** *The Navier-Stokes equation can be encoded on the affine fiber bundle  $\pi : W \equiv M \times \mathbf{I} \times \mathbb{R}^2 \rightarrow M$ ,  $(x^\alpha, \dot{x}^i, p, \theta)_{0 \leq \alpha \leq 3, 1 \leq i \leq 3} \mapsto (x^\alpha)$ . (See [22].) Therefore, Cauchy manifolds are 3-dimensional manifolds. For such dimension do not exist exotic spheres. Therefore, the Navier-Stokes equation cannot be an exotic PDE. Similar considerations hold for PDE's of the classical continuum mechanics.*

**Example 2.11.** *Let  $M$  be a  $n$ -dimensional manifold,  $n \geq 2$ , and let  $\pi : E \equiv M \times \mathbb{R} \rightarrow M$  be the trivial vector fiber bundle on  $M$ . The  $n$ -d'Alembert equation:<sup>6</sup>  $\frac{\partial^n \log f}{\partial x_1 \dots \partial x_n} = 0$ , is a  $n$ -order closed partial differential relation (in the sense of Gromov [9]) on the fiber bundle  $\pi : E \equiv M \times \mathbb{R} \rightarrow M$ , i.e., it defines a subset  $Z_n \subset JD^n(E)$ , without boundary,  $\partial Z_n = \emptyset$ . Let  $\{x^\alpha, u, u_\alpha, u_{\alpha\beta}, \dots, u_{\alpha_1 \dots \alpha_n}\}$  be a coordinate system on  $JD^n(E)$  adapted to the fiber structures:  $\pi_n : JD^n(E) \rightarrow M$ ,  $\bar{\pi}_{n,0} : JD^n(E) \rightarrow \mathbb{R}$ . Then,  $Z_n = F^{-1}(0)$ ,  $F : JD^n(E) \rightarrow \mathbb{R}$ , where  $F$  is a sum of terms of the type:*

$$F[s; r | \alpha, \beta_1 \beta_2, \dots, \gamma_1 \dots \gamma_q] \equiv su^r u_\alpha u_{\beta_1 \beta_2} \dots u_{\gamma_1 \dots \gamma_q},$$

with  $\alpha \neq \beta_1 \neq \beta_2 \neq \dots \neq \gamma_1 \neq \dots \neq \gamma_q \leq n$ ,  $s \in \mathbb{Z}$ ,  $r \in \mathbb{N} \cup \{0\}$ . Furthermore, the term in  $F$  containing  $u_{1\dots n}$  is just  $u_{1\dots n} u^{n-1}$ .<sup>7</sup> Note that  $F$  has not locally constant rank on all  $Z_n$ , so  $Z_n$  is not a submanifold of  $JD^n(E)$ . Furthermore, on the open subset  $C_n \equiv u^{-1}(\mathbb{R} \setminus 0) \subset JD^n(E)$ , one recognizes that  $F$  has locally constant rank 1. Hence  $Z_n \cap C_n$  is a subbundle of  $JD^n(E) \rightarrow M$ , of dimension  $n + \frac{(2n)!}{(n!)^2} - 1$ . In the following, for abuse of notation, we shall denote by  $(d'A)_n$  whether  $Z_n$  or  $Z_n \cap C_n$ .

The  $n$ -d'Alembert equation over  $M = \mathbb{R}^n$  can be an exotic PDE for  $n$ -dimensional manifolds of dimension  $n \geq 8$ , but one must carefully consider the meaning of smooth Cauchy  $(n-1)$ -dimensional manifolds there. In fact, it is not possible for any  $n$  to embed in the fiber bundle  $E = \mathbb{R}^{n+1}$  exotic  $(n-1)$ -spheres. To be more precise, let us consider the case  $n = 8$ . Then since  $S^7 \subset \mathbb{R}^8 \subset E$ , we can embed in  $E$  the standard 7-dimensional sphere. On the other hand it is well known, after some results by E. V. Brieskorn [4, 5], that homotopy 7-spheres  $\Sigma^7$  can be identified with the intersections of complex hypersurfaces  $Y_\kappa$ ,  $1 \leq \kappa \leq 28$ , in  $\mathbb{C}^5$ , with a 9-dimensional small sphere  $X$ , around the singular point at the origin in  $Y_\kappa$ :  $\Sigma^7 = Y_\kappa \cap X$ . See equations in (1).

$$(1) \quad \Sigma^7 = \left\{ (z_1, \dots, z_5) \in \mathbb{C}^5 \mid \begin{array}{l} (Y_\kappa) : z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^{6\kappa-1} = 0, \quad 1 \leq \kappa \leq 28 \\ (X) : \sum_j (z_j \bar{z}_j) = 1 \end{array} \right\} \subset \mathbb{C}^5 \cong \mathbb{R}^{10}.$$

$X \cap Y_\kappa$ ,  $1 \leq \kappa \leq 28$ , have the differential structures identified by  $\Theta_7 \cong \mathbb{Z}_{28}$ .<sup>8</sup> In other words exotic 7-spheres are framed manifolds  $\Sigma^7 \subset \mathbb{R}^{7+s}$ , with  $s \geq 3$ . Therefore, we cannot embed in the total space  $E \equiv \mathbb{R}^9$ , of the fiber bundle  $\pi : E \rightarrow \mathbb{R}^8$ , any homotopy 7-sphere. However, this does not exclude that some smooth Cauchy 7-dimensional manifolds in  $(d'A)_8$  can be identified with exotic 7-spheres. In fact, since  $\dim(d'A)_8 = 12877$ , it is satisfied the (Whitney) condition  $\dim(d'A)_8 \geq$

<sup>6</sup>If  $n = 2$  we say simply d'Alembert equation and we will put  $(d'A) \equiv (d'A)_2$ .

<sup>7</sup>For example, for  $n = 2$  one has:  $F = u_{xy}u - u_x u_y$ , and for  $n = 3$  one has:  $F = u_{xyz}u^2 - u_{xy}u_z u - u_{xz}u_y u + u_x u_y u_z$ .

<sup>8</sup> $\Theta_n$  denotes the additive group of diffeomorphism classes of oriented smooth homotopy spheres of dimension  $n$ .

$2 \times 7 + 1 = 15$  to embed  $\Sigma^7$  in  $(d'A)_8$ . If  $N \subset (d'A)_8$  is the image of such an embedding,  $N$  cannot in general be diffeomorphic to its image  $Y \subset E$ , via the canonical projection  $\pi_{8,0} : JD^8(E) \rightarrow E$ . So, in this case we shall talk about singular Cauchy 7-manifolds of  $(d'A)_8$ . Furthermore, let us emphasize that since the equation defining the open PDE  $C_8 \cap (d'A)_8$ , can be solved with respect to the coordinate  $u_{x^1 \dots x^n}$ , we can embed homotopy 7-spheres  $\Sigma^7$  as smooth integral submanifolds  $N \subset (d'A)_8 \subset J_8^8(E)$ , such that their Thom-Boardman singular points should be not frozen singularities in the sense introduced in [13]. Therefore, we can state that such 7-dimensional integral manifolds are contained in 8-dimensional integral manifolds  $V \subset (d'A)_8$ , (singular) solutions of  $(d'A)$ . Such 7-dimensional integral manifolds are called admissible Cauchy manifolds of  $(d'A)_8$ .

### 3. STABILITY IN EXOTIC $n$ -D'ALEMBERT PDE's

Let us consider, now, the stability of PDE's in the framework of the geometric theory of PDE's. We shall follow the line just drawn in some of our previous papers on this subject, where we have unified the integral bordism for PDE's and stability and related the quantum bordism of PDE's to Ulam stability [46].

**Definition 3.1.** (Singular solutions of PDE's). Let  $\pi : W \rightarrow M$  be a fiber bundle with  $\dim M = n$  and  $\dim W = m + n$ . Let  $E_k \subset JD^k(W)$  be a PDE and  $V \subset E_k$  be a solution of  $E_k$ . We say that  $p \in V$  is a singular point of  $V$  of type  $\Sigma_i$ ,  $i = 0, 1, \dots, n$ , if the canonical map  $\pi_k|_V : V \rightarrow M$  has a Thom-Boardman singularity of type  $S_i$  [3, 29]. Let  $\Sigma(V) \subset V$  be the set of singular points of  $V$ . Then  $V \setminus \Sigma(V) = \bigcup_r V_r$  is the disjoint union of connected components  $V_r$ . For every of such components  $\pi_k : V_r \rightarrow M$  is an immersion and can be represented by means of image of  $k$  derivative of some section  $s$  of  $\pi$ , i.e.,  $V_r = D^k s(U_r)$ , where  $U_r \subset M$  is an open subset of  $M$ . We call also solutions of  $E_k$  any submanifold  $V \subset E_k$  that are obtained by projection of  $\pi_{k+h,k}$  of some solution  $V' \subset (E_k)_{+h} \subset JD^{k+h}(W)$ , represented by a smooth integral submanifold of  $(E_k)_{+h}$ , i.e.,  $V = \pi_{k+h,k}(V')$ . In general a such a solution  $V$  is not more represented by a smooth submanifold of  $E_k$ . We say also that instead the smooth manifold  $V' \subset (E_k)_{+h}$  solves singularities of  $V$ , (or smooths  $V$ ). More general solutions are considered taking into account the canonical embedding  $JD^k(W) \hookrightarrow J_n^k(W)$ , where  $J_n^k(W)$  is the  $k$ -jet space for  $n$ -dimensional submanifolds of  $W$ . (For details see [20, 21, 22, 23, 25, 26].)

We define weak solutions, solutions  $V \subset E_k$ , such that the set  $\Sigma(V)$  of singular points of  $V$ , contains also discontinuity points,  $q, q' \in V$ , with  $\pi_{k,0}(q) = \pi_{k,0}(q') = a \in W$ , or  $\pi_k(q) = \pi_k(q') = p \in M$ . We denote such a set by  $\Sigma(V)_S \subset \Sigma(V)$ , and, in such cases we shall talk more precisely of singular boundary of  $V$ , like  $(\partial V)_S = \partial V \setminus \Sigma(V)_S$ . However for abuse of notation we shall denote  $(\partial V)_S$ , (resp.  $\Sigma(V)_S$ ), simply by  $(\partial V)$ , (resp.  $\Sigma(V)$ ), also if no confusion can arise.

**Definition 3.2.** (Stable solutions of PDE's). Under the same hypotheses of above definitions, let  $X \rightarrow E_k$  be a regular solution, where  $X \subset M$  is a smooth  $n$ -dimensional compact manifold with boundary  $\partial X$ . Then  $f$  is stable if there is a neighborhood  $W_f$  of  $f$  in  $\text{Sol}(E_k)$ , the manifold of regular solutions of  $E_k$ , such that each  $f' \in W_f$  is equivalent to  $f$ , i.e.,  $f$  is transformed in  $f'$  by vertical symmetries of  $E_k$ .

**Theorem 3.3.** [29] Let  $E_k \subset JD^k(W)$  be a  $k$ -order PDE on the fiber bundle  $\pi : W \rightarrow M$  in the category of smooth manifolds,  $\dim W = m + n$ ,  $\dim M = n$ ,

$m > 1$ . Let  $s : M \rightarrow W$  be a section, solution of  $E_k$ , and let  $\nu : M \rightarrow s^*vTW \equiv E[s]$  be a solution of the linearized equation  $E_k[s] \subset JD^k(E[s])$ . Then to  $\nu$  is associated a flow  $\{\phi_\lambda\}_{\lambda \in J}$ , where  $J \subset \mathbb{R}$  is a neighborhood of  $0 \in \mathbb{R}$ , that transforms  $V$  into a new solution  $\tilde{V} \subset E_k$ .

**Definition 3.4.** Let  $E_k \subset J_n^k(W)$ , where  $\pi : W \rightarrow M$  is a fiber bundle, in the category of smooth manifolds. We say that  $E_k$  is functionally stable if for any compact regular solution  $V \subset E_k$ , such that  $\partial V = N_0 \cup P \cup N_1$  one has quantum solutions  $\tilde{V} \subset J_n^{k+s}(W)$ ,  $s \geq 0$ , such that  $\pi_{k+s,0}(\tilde{N}_0 \sqcup \tilde{N}_1) = \pi_{k,0}(N_0 \sqcup N_1) \equiv X \subset W$ , where  $\partial \tilde{V} = \tilde{N}_0 \cup \tilde{P} \cup \tilde{N}_1$ .

We call the set  $\Omega[V]$  of such solutions  $\tilde{V}$  the full quantum situs of  $V$ . We call also each element  $\tilde{V} \in \Omega[V]$  a quantum fluctuation of  $V$ .<sup>9</sup>

We call infinitesimal bordism of a regular solution  $V \subset E_k \subset JD^k(W)$  an element  $\tilde{V} \in \Omega[V]$ , defined in the proof of Theorem 3.3. (See [29].) We denote by  $\Omega_0[V] \subset \Omega[V]$  the set of infinitesimal bordisms of  $V$ . We call  $\Omega_0[V]$  the infinitesimal situs of  $V$ .

**Definition 3.5.** Let  $E_k \subset J_n^k(W)$ , where  $\pi : W \rightarrow M$  is a fiber bundle, in the category of smooth manifolds. We say that a regular solution  $V \subset E_k$ ,  $\partial V = N_0 \cup P \cup N_1$ , is functionally stable if the infinitesimal situs  $\Omega_0[V] \subset \Omega[V]$  of  $V$  does not contain singular infinitesimal bordisms.

**Theorem 3.6.** [28, 29] Let  $E_k \subset J_n^k(W)$ , where  $\pi : W \rightarrow M$  is a fiber bundle, in the category of smooth manifolds. If  $E_k$  is formally integrable and completely integrable, then it is functionally stable as well as Ulam-extended superstable. A regular solution  $V \subset E_k$  is stable iff it is functionally stable.

**Remark 3.7.** Let us emphasize that the definition of functionally stable PDE interprets in pure geometric way the definition of Ulam superstable functional equation just adapted to PDE's.

**Definition 3.8.** We say that  $E_k \subset JD^k(W)$  is a stable extended crystal PDE if it is an extended crystal PDE that is functionally stable and all its regular smooth solutions are (functionally) stable.

We say that  $E_k \subset JD^k(W)$  is a stabilizable extended crystal PDE if it is an extended crystal PDE and to  $E_k$  can be canonically associated a stable extended crystal PDE  ${}^{(S)}E_k \subset JD^{k+s}(W)$ . We call  ${}^{(S)}E_k$  just the stable extended crystal PDE of  $E_k$ .

We have the following criteria for functional stability of solutions of PDE's and to identify stable extended crystal PDE's.

**Theorem 3.9.** (Functional stability criteria).[29] Let  $E_k \subset JD^k(W)$  be a  $k$ -order formally integrable and completely integrable PDE on the fiber bundle  $\pi : W \rightarrow M$ ,  $\dim W = m + n$ ,  $\dim M = n$ .

1) If the symbol  $g_k = 0$ , then all the smooth regular solutions  $V \subset E_k \subset JD^k(W)$  are functionally stable, with respect to any non-weak perturbation. So  $E_k$  is a stable extended crystal.

<sup>9</sup>Let us emphasize that to  $\Omega[V]$  belong also (non necessarily regular) solutions  $V' \subset E_k$  such that  $N'_0 \sqcup N'_1 = N_0 \sqcup N_1$ , where  $\partial V' = N'_0 \cup P' \cup N'_1$ .



- 2) If  $E_k$  is of finite type, i.e.,  $g_{k+r} = 0$ , for  $r > 0$ , then all the smooth regular solutions  $V \subset E_{k+r} \subset JD^{k+r}(W)$  are functionally stable, with respect to any non-weak perturbation. So  $E_k$  is a stabilizable extended crystal with stable extended crystal  ${}^{(S)}E_k = E_{k+r}$ .
- 3) If  $V \subset (E_k)_{+\infty} \subset JD^\infty(W)$  is a smooth regular solution, then  $V$  is functionally stable, with respect to any non-weak perturbation. So any formally integrable and completely integrable PDE  $E_k \subset JD^k(W)$ , is a stabilizable extended crystal, with stable extended crystal  ${}^{(S)}E_k = (E_k)_{+\infty}$ .

**Remark 3.10.** Let us also remark that in evolutionary PDE's, i.e., PDE's built on a fiber bundle  $\pi : W \rightarrow M$ , over a "space-time"  $M$ ,  $\{x^\alpha, y^j\}_{0 \leq \alpha \leq n, 1 \leq j \leq m} \mapsto \{x^\alpha\}_{0 \leq \alpha \leq n}$ , where  $x^0 = t$  represents the time coordinate, one can consider "asymptotic stability", i.e., the behaviour of perturbations of global solutions for  $t \rightarrow \infty$ . In such cases we can recast our formulation on the corresponding compactified space-times. (For details see [29, 30].)

From above results one can see that, in general, the functional stability of smooth regular solutions is a very strong requirement. However, above theorems, give us workable criteria to obtain subequations of  $E_k$  whose smooth regular solutions have assured functional stability.

It is also useful an weaker requirement than functional stability. This is related to a concept of "averaged stability".

In fact we have the following definition.

**Definition 3.11.** Let  $E_k \subset JD^k(W)$  be a formally integrable and completely integrable PDE the fiber bundle  $\pi : W \rightarrow M$ , and let  $V = D^k s(M) \subset E_k$  be a regular smooth solution of  $E_k$ . Let  $\xi : M \rightarrow E_k[s]$  be the general solution of  $E_k[s]$ . Let us assume that there is an Euclidean structure on the fiber of  $E[s] \rightarrow M$ . Then, we say that  $V$  is average asymptotic stable if the function of time  $\mathfrak{p}(t)$  defined by the formula:

$$(2) \quad \mathfrak{p}(t) = \frac{1}{2\text{vol}(B_t)} \int_{B_t} \xi^2 \eta$$

has the following behaviour:  $\mathfrak{p}(t) = \mathfrak{p}(0)e^{-ct}$  for some real number  $c > 0$ . We call  $\tau_0 = 1/c_0$  the characteristic stability time of the solution  $V$ . If  $\tau_0 = \infty$  it means that  $V$  is average instable.<sup>10</sup>

We have the following criterion of average asymptotic stability.

**Theorem 3.12.** (Criterion of average asymptotic stability).[29] A regular global smooth solution  $s$  of  $E_k$  is average stable if the following conditions are satisfied:

$$(3) \quad \dot{\mathfrak{p}}(t) \leq c \mathfrak{p}(t), \quad c \in \mathbb{R}^+, \forall t.$$

where

$$(4) \quad \dot{\mathfrak{p}}(t) = \frac{1}{2\text{vol}(B_t)} \int_{B_t} \left( \frac{\delta \xi^2}{\delta t} \right) \eta = \frac{1}{\text{vol}(B_t)} \int_{B_t} \left( \frac{\delta \xi}{\delta t} \cdot \xi \right) \eta.$$

Here  $\xi$  represents the general solution of the linearized equation  $E_k[s]$  of  $E_k$  at the solution  $s$ . Let us denote by  $c_0$  the infimum of the positive constants  $c$  such that

<sup>10</sup>In the following, if there are not reasons of confusion, we shall call also stable solution a smooth regular solution of a PDE  $E_k \subset JD^k(W)$  that is average asymptotic stable.

inequality (3) is satisfied. Then we call  $\tau_0 = 1/c_0$  the characteristic stability time of the solution  $V$ . If  $\tau_0 = \infty$  means that  $V$  is unstable.<sup>11</sup> Furthermore, condition (3) is satisfied if the operator  $\frac{\delta}{\delta t}$  is self-adjoint on the set of solutions of the linearized equation  $E_k[s] \subset J_n^n(E[s])$ , where  $E[s] \equiv s^*vTW$ .

**Theorem 3.13.** (The extended crystal structure of the  $n$ -d'Alembert equation, and stability).[31] 1) For the  $n$ -d'Alembert equation one has the following properties.

(i) The  $n$ -d'Alembert equation is an extended crystal PDE for any  $n \geq 2$ . If  $M$  is  $p$ -connected,  $p \in \{0, 1, \dots, n-1\}$ , it becomes an extended 0-crystal iff  $\Omega_{n-1} = 0$ . In particular for  $n = 2$  it becomes a 0-crystal.

(ii) The  $n$ -d'Alembert equation is functionally stable.

(iii) Smooth regular solutions of the  $n$ -d'Alembert equation, present, in general, unstabilities at finite times. However, the  $n$ -d'Alembert equation can be stabilized and its stable extended crystal PDE is its  $\infty$ -prolongation  $((d'A)_n)_{+\infty}$ . There all smooth regular solutions are functionally stable, i.e., they do not present finite times unstabilities.

2) In the case  $n = 2$ , with  $M$  non-simply connected,  $(d'A)$  remains an extended crystal PDE, but not more an extended 0-crystal PDE. For example if  $M$  is a bidimensional torus  $T^2$ . This is a connected, orientable, non-simply connected, surface. Then,  $\Omega_1^{(d'A)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$  (For a proof see [31].) So, the d'Alembert equation on the torus is not an extended 0-crystal PDE, and neither an a 0-crystal PDE. The crystal group of such an equation is  $G(2) = \mathbb{Z} \rtimes D_4 = p4m$ . Its crystal dimension is 2.

In the case  $n = 2$ , with  $M = \mathbb{R}^2$ , we can built solutions with the methods of characteristics, that are average unstable.

3) Let us consider the 3-d'Alembert equation on the non-simply connected, orientable, 3-dimensional manifold  $M = \mathbb{R}P^3$ . In this case one has  $\Omega_2^{(d'A)} \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Thus this is another example where one has  $(d'A)_3$  that is an extended crystal PDE, but it cannot be an extended 0-crystal PDE and neither a 0-crystal PDE. Thus this equation has the same crystal group and crystal dimension of equation considered in above example.

*Proof.* Even if these results are proved in [31], let us resume their proofs here, in order to better understand the following ones.

1) (i) The  $n$ -d'Alembert equation  $(d'A)_n \subset JD^n(E)$  is a  $n$ -order PDE, formally integrable, and completely integrable, on the trivial vector fiber bundle  $\pi : E \equiv M \times \mathbb{R} \rightarrow M$ .<sup>12</sup> (See [41]). This means that we can locally reproduce all the results obtained for the  $n$ -D'Alembert equation on  $\mathbb{R}^n$ . (See Refs.[22, 41]). For any point  $q \in (d'A)_n$ , passes a local solution. Furthermore, the set of local solutions of the  $n$ -d'Alembert equation on  $n$ -dimensional manifolds contains the set of the local functions that can be represented in the form as  $f(x^1, \dots, x^n) = f_1(x^2, \dots, x^n) \cdots f_n(x^1, \dots, x^{n-1})$ . This follows directly from previous considerations and results contained in refs.[22, 40, 41]. Now, the set  $\mathfrak{Sol}_{loc}(d'A)_n$ ,  $n \geq 2$ , of all local solutions of the equation:  $\frac{\partial^n \log f}{\partial x_n \cdots \partial x_1} = 0$ , considered on a  $n$ -dimensional manifold  $M$ , is larger than the set of all local functions  $f$  that can be represented in the form  $f(x^1, \dots, x^n) = f_1(x^2, \dots, x^n) \cdots f_n(x^1, \dots, x^{n-1})$ . (See [40, 41].) In the

<sup>11</sup> $\tau_0$  has just the physical dimension of a time.

<sup>12</sup> $(d'A)_n$  considered in this theorem is a submanifold of  $JD^n(E)$ , hence it coincides with  $Z_n \cap C_n$ .



following we shall consider the  $n$ -d'Alembert equation given as a submanifold  $(d'A)_n$  of the jet space  $J_n^n(E)$  by means of the embedding  $(d'A)_n \hookrightarrow JD^n(E) \hookrightarrow J_n^n(E)$ . The characterization of global solutions of  $(d'A)_n$  is made by means of its integral bordism groups. One has:  $\Omega_p^{(d'A)_n} \cong \Omega_p((d'A)_n)$ , for  $p \in \{0, \dots, n-1\}$ . This follows from the fact that the  $n$ -d'Alembert equation is formally integrable, and completely integrable. (See [41].) We get:  $\Omega_p^{(d'A)_n} \cong \underline{\Omega}_p(M) \cong \bigoplus_{r,s,r+s=p} H_r(M; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} \Omega_s$ ,  $p \in \{0, \dots, n-1\}$ . In the particular case that  $\dim M = 2$  and  $p$ -connected,  $p \in \{0, 1\}$ , then the integral bordism group  $\Omega_1^{(d'A)} = 0$ . Thus  $(d'A)$  is an extended 0-crystal PDE. Furthermore, one can also prove that for such a case there are not obstructions coming from the integral characteristic numbers. In fact, all the conservation laws on closed 1-dimensional smooth integral manifolds are zero [22]. Then one has  $\text{cry}(d'A) = 0$ , for  $p$ -connected  $M$ ,  $p \in \{0, 1\}$ . Thus in this case  $(d'A)$  becomes a 0-crystal.

(ii) The  $n$ -d'Alembert equation is functionally stable since it is formally integrable and completely integrable. (See Theorem 3.6.)

(iii) The functional unstabilities come from the fact that the symbol of the  $n$ -d'Alembert equation is not zero. In fact one has

$$(5) \quad \dim(g_n)_q = \frac{(2n-1)!}{n!(n-1)!} - 1, \quad \forall q \in (d'A)_n.$$

Furthermore, in the  $\infty$ -prolongation  $((d'A)_n)_{+\infty} \subset J_n^\infty(W)$ , we get all the smooth solutions of  $(d'A)_n$ , and there, since the corresponding symbol is zero,  $((g_n))_{+\infty} = 0$ , do not exist admissible singular (non-weak) perturbations. Thus,  $((d'A)_n)_{+\infty}$  is necessarily the stable extended crystal of  $(d'A)_n$ . Therefore,  $(d'A)_n$  is a stabilizable PDE.

2) We have proved in Refs.[40, 41] that  $(d'A)$  admits the following characteristics strips:

$$(6) \quad \begin{cases} \zeta_1 & \equiv u[\partial y + u_y \partial u + u_{yx} \partial u_x + u_{yy} \partial u_y] + u_{xx} u_y \partial u_{xx} + u_{yy} u_x \partial u_{yy} \\ \zeta_2 & \equiv u[\partial x + u_x \partial u + u_{yx} \partial u_y + u_{xx} \partial u_x] + u_{xx} u_y \partial u_{xy} + u_{yy} u_x \partial u_{yy}. \end{cases}$$

These generate characteristic 1-dimensional distributions respectively in the following sub-equations  $_i(d'A) \subset (d'A)$ ,  $i = 1, 2$ :

$$(7) \quad {}_1(d'A) : \left\{ \begin{array}{l} u_{xx} = 0 \\ uu_{xy} - u_x u_y = 0 \end{array} \right\}; \quad {}_2(d'A) : \left\{ \begin{array}{l} u_{yy} = 0 \\ uu_{xy} - u_x u_y = 0 \end{array} \right\}.$$

For such equations the above mentioned 1-dimensional distributions are respectively characteristic distribution. Therefore, for such equations we can built characteristic solutions that are, of course, also solutions of  $(d'A)$ . For example, we have proved in [41] that the solution generated by  $\zeta_1$  is given by the following formula:

$$(8) \quad u(x, y) = \left(\frac{\beta}{2}y^2 + \alpha y + 1\right)h(x),$$

where  $\alpha, \beta \in \mathbb{R}$  and  $h(x)$  is an arbitrary function on one real variable. Let us, now, investigate, if such a solution is average stable. The parametric equations for the characteristic flow on such a solution, say  $V \subset (d'A)$ , is given by the following differential system:

$$(9) \quad \begin{cases} \dot{x} = 0 \\ \dot{y} = u \\ \dot{u} = uu_y. \end{cases}$$

The general solution of the linearized equation  $(d'A)[V] \subset JD^2(E[s])$  can be obtained from the general symmetry vector field for  $(d'A)$ , given in [41]. Then we get

$$(10) \quad \begin{cases} \xi = [s(y) + r(x)]u\partial u \\ u(x, y) = (\frac{\beta}{2}y^2 + \alpha y + 1)h(x), \end{cases}$$

where  $s$  and  $r$  are arbitrary functions. Let us denote by  $\xi(x, y)$  the component of the vertical vector field  $\xi$ . Then one explicitly has  $\xi(x, y) = [s(y) + r(x)](\frac{\beta}{2}y^2 + \alpha y + 1)h(x)$ . For the arbitrariness of the functions  $r$ ,  $s$  and  $h$ , one can see that  $\xi(x, y)$  can have singular points. So the solution (8) is not stable in  $(d'A)$ . Furthermore, it is not asymptotic stable, since  $\lim_{y \rightarrow \infty} \xi(x, y) = \infty$ . In order to investigate whether it is average stable, let us consider the differential operator  $\frac{\delta \xi}{\delta t}$  on  $(d'A)[V]$ . One has  $\frac{\delta \xi}{\delta t} = (\partial_t \xi) + (\partial_x \xi)\dot{x} + (\partial_y \xi)\dot{y} = (\partial_y \xi)u(x, y)$ . For its adjoint, one has  $\frac{\delta^* \phi}{\delta t} = -(\partial_y \cdot (u(x, y)\phi)) = -(\partial_y \phi)u(x, y) - (\partial_y u(x, y))\phi$ . Thus, the operator  $\frac{\delta \xi}{\delta t}$  is not self-adjoint on the solution in (8), hence, such a solution is not average stable.

3) This follows directly from previous points.  $\square$

**Theorem 3.14** (Stability in exotic 8-d'Alembert PDE's over  $\mathbb{R}^8$ ). *Let us consider  $(d'A)_8$  over  $\mathbb{R}^8$ . The integral singular bordism group  $\Omega_{7,s}^{(d'A)_8}$  of the 8-d'Alembert PDE over  $\mathbb{R}^8$  is  $\Omega_{7,s}^{(d'A)_8} = \mathbb{Z}_2$ . If we consider admissible Cauchy manifolds  $N \subset (d'A)_8$ , identified with 7-dimensional homotopy spheres, (homotopy equivalence full admissibility hypothesis), then one has  $\Omega_{7,s}^{(d'A)_8} \cong \Omega_7^{(d'A)_8} = 0$ , hence  $(d'A)_8$  becomes an extended 0-crystal PDE, but also a 0-crystal PDE. The bordism classes in  $\Omega_7^{(d'A)_8}$  are identified by Cauchy manifolds represented by diffeomorphic homotopy spheres. In particular, in the homotopy equivalence full admissibility hypothesis, starting from an admissible Cauchy manifold  $N_0 \subset (d'A)_8$ , identified with  $S^7$ , one can arrive with a singular solution, to any other admissible Cauchy manifold  $N_1 \subset (d'A)_8$ . Such a solution is unstable. Moreover, there exists a smooth solution  $V$ , such that  $V = N_0 \sqcup N_1$ , iff  $N_0 \cong N_1$ . Such a solution can be stabilized.*

*Proof.* In fact,  $\Omega_{7,s}^{(d'A)_8} \cong \bigoplus_{0 \leq r, s \leq 7} \Omega_r \otimes_{\mathbb{Z}_2} H_s(M; \mathbb{Z}_2)$ . Taking into account that for  $M \cong \mathbb{R}^8$  one has  $H_r(M; \mathbb{Z}_2) = 0$  for  $0 < r \leq 7$ , and  $H_0(M; \mathbb{Z}_2) = \mathbb{Z}_2$ , and that  $\Omega_7 \cong \mathbb{Z}_2$ , we get  $\Omega_{7,s}^{(d'A)_8} = \mathbb{Z}_2$ . If we consider admissible Cauchy 7-dimensional homotopy spheres only, we have that they have necessarily all integral characteristic numbers, i.e., the evaluations on such manifolds of all the conservation laws, give same numbers. (For the proof one can copy the similar proof given in [38, 1] for the Ricci flow equation.) Therefore they belong to the same singular integral bordism class, i.e.  $\Omega_{7,s}^{(d'A)_8} = 0$ . Since, one has the short exact sequence (11),

$$(11) \quad 0 \longrightarrow K_{7,s}^{(d'A)_8} \longrightarrow \Omega_7^{(d'A)_8} \longrightarrow \Omega_{7,s}^{(d'A)_8} \longrightarrow 0.$$

we get that under the *homotopy equivalence full admissibility hypothesis*, one has  $\Omega_7^{(d'A)_8} \cong K_{7,s}^{(d'A)_8}$ . Let us emphasize that even if the number of differentiable structures on 7-dimensional spheres is 28, smooth Cauchy-manifolds-exotic-7-spheres cannot be contained into  $((d'A)_8)_{+\infty}$ , since they are singular integral manifolds. So smooth Cauchy manifolds contained into  $((d'A)_8)_{+\infty}$  can be identified with  $S^7$  only. Furthermore, taking into account that smooth solutions, bording smooth

Cauchy manifolds, necessitate to identify diffeomorphisms between the corresponding sectional submanifolds, it follows that must be  $\Omega_7^{(d'A)_8} = 0$  too. Therefore, we get also  $\text{cry}((d'A)_8) = 0$ .

For the previous arguments it is important to state that the space of conservation laws is not zero.

**Lemma 3.15.** *The space of conservation laws of  $(d'A)_n$  is not zero.*

*Proof.* In fact, conservation laws of  $(d'A)_n$  are  $(n-1)$ -differential forms  $\omega = \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$  on  $((d'A)_n)_{+\infty}$ , such that for any smooth integral  $n$ -manifold  $V \subset (d'A)_n$ , solution of  $(d'A)_n$ , one has  $d\omega|_V = 0$ . Then, one can take the  $(n-1)$ -differential forms  $\omega$ , given in (12).

$$(12) \quad \begin{cases} \omega &= \omega_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ \omega_i &= \omega(x^1, \dots, \widehat{x^i}, \dots, x^n, I_{\alpha i})_{\alpha \neq i, |\alpha| \geq 0} \\ I_{\alpha i} &\equiv (\partial x_{\alpha} \cdot (\partial x_1 \cdots \partial x_n \cdot \log f)) \end{cases}$$

where  $f : M \rightarrow \mathbb{R}$  is a smooth function on  $M$  and  $\omega_i$  are arbitrary smooth functions of their arguments. The "widehat" over the symbols means absence of the underlying symbols. In fact, one has the following.

$$(13) \quad \begin{cases} d\omega &= \sum_{1 \leq i \leq n} \sum_{p \neq i} (\partial x_p \cdot \omega_i) dx^p \wedge dx^1 \wedge \cdots \wedge \widehat{dx^i} \cdots \wedge dx^n = 0 \\ &+ \sum_{1 \leq i \leq n} \sum_{\alpha \neq i, |\alpha| \geq 0} (\partial I_{i \alpha} \cdot \omega_i) (\partial x_i \cdot I_{i \alpha}) dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_{\alpha \neq i, |\alpha| \geq 0} [\sum_{1 \leq i \leq n} (\partial I_{i \alpha} \cdot \omega_i)] (\partial x_{\alpha} \cdot (\partial x_1 \cdots \partial x_n \cdot \log f)) dx^1 \wedge \cdots \wedge dx^n \end{cases}$$

Now  $d\omega|_V = 0$  if  $V \subset (d'A)_n$  is a smooth  $(n+1)$ -dimensional integral manifold, (singular) solution of  $(d'A)_n$ . In fact, if  $f$  satisfies the equation  $(\partial x_1 \cdots \partial x_n \cdot \log f) = 0$ , then  $(\partial x_{\alpha} \cdot (\partial x_1 \cdots \partial x_n \cdot \log f)) = 0$ , for  $|\alpha| \geq 0$ . This directly follows from the prolongations of the formally integrable and completely integrable  $n$ -d'Alembert equation. For example for  $n = 2$  we get for  $(d'A)_2$  and its first prolongation  $((d'A)_2)_{+1}$  the equations given in (14).

$$(14) \quad (d'A)_2 : \{f_{xy}f - f_x f_y = 0\}; ((d'A)_2)_{+1} : \left\{ \begin{array}{l} f_{xy}f - f_x f_y = 0 \\ f_{xxy}f - f_{xx}f_y = 0 \\ f_{xyy}f - f_{yy}f_x = 0 \end{array} \right\}.$$

One other hand we have:

$$(15) \quad \begin{cases} (\partial x(\partial x \partial y \cdot \log f)) &= (f_{xxy}f - f_{xx}f_y)f - 2(f_{xy}f - f_x f_y)f_x \\ (\partial y(\partial x \partial y \cdot \log f)) &= (f_{xyy}f - f_{yy}f_x)f - 2(f_{xy}f - f_x f_y)f_y \end{cases}$$

Therefore, on the 2-d'Alembert equation one has

$$(16) \quad \begin{cases} (\partial x(\partial x \partial y \cdot \log f))|_V = 0 \\ (\partial y(\partial x \partial y \cdot \log f))|_V = 0. \end{cases}$$

This process can be iterated on all the prolongation orders.  $\square$

To conclude the proof of Theorem 3.14 it is enough to consider that at finite order, where live singular solutions, the symbol of the 8-d'Alembert equation is not zero. Thus these solution are unstable. Instead, smooth solutions can be stabilized, since these can be identified with smooth integral manifolds of the infinity prolongation  $((d'A)_8)_{+\infty}$ , where the symbol is zero. (See Theorem 3.9.)  $\square$

**Corollary 3.16.** *In the homotopy equivalence full admissibility hypothesis,  $(d'A)_8$  admits a singular global attractor, in the sense introduced in [37], i.e., all the admissible Cauchy manifolds belong to the same integral singular bordism class of  $(d'A)_8$ . Furthermore, in the sphere full admissibility hypothesis, i.e. when we consider admissible all the smooth Cauchy manifolds identifiable via diffeomorphisms with  $S^7$ ,  $(d'A)_8$  admits a smooth global attractor, in the sense that all the smooth admissible Cauchy manifolds belong to the same integral smooth bordism class of  $(d'A)_8$ .*

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